Appendix D

Spectral Resolution and Sensitivity
Bandwidth in Spectrometers

For many of the spectrometers used at millimeter-wavelength observatories (filter bank spectrometers, correlation spectrometers, etc.), the true bandwidth of a single spectrometer channel is a convolution of the original gain response of the channel and any smoothing functions applied. This convolution of the response and smoothing functions affects two important factors:

\[ \Delta v_{FWHP} \equiv \text{The full-width to half-power, or -3dB, bandwidth, which is the true “spectral resolution”, and} \]
\[ \Delta v_{SB} \equiv \text{The “sensitivity bandwidth”, which is the channel bandwidth used in calculations of the rms sensitivity in a spectrum.} \]

The sensitivity bandwidth is defined as follows:

\[
\Delta v_{SB} \equiv \frac{\int_{-\infty}^{\infty} F(v) \, dv}{\int_{-\infty}^{\infty} [F(v)]^2 \, dv}
\]


Table D.1 describes the relationship between \( \Delta v_{FWHP} \), \( \Delta v_{SB} \), and the frequency sampling \( \Delta v \) for various smoothing functions \( F(v) \) assuming Nyquist sampling at \( \Delta v \). At the end of this Appendix I show the calculation for each of these integrals.

For the filter bank spectrometers at the 12m, the channel response is a second order Chebyshev bandpass filter (which approximates a pill box function), there is no smoothing, and they are not Nyquist sampled. A single channel in the Millimeter Autocorrelator (MAC), on the other hand, has a sinc frequency response which is hanning smoothed. Since the convolution of a sinc with any function that is already band-limited withing the frequency response of the sinc leaves that function unchanged, we are left with a hanning function response. Therefore:
Table D.1: Spectral Resolution and Sensitivity Bandwidth

<table>
<thead>
<tr>
<th>Function</th>
<th>$\Delta v_{FWHP}^{a}$</th>
<th>$\Delta v_{SB}^{a}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>sinc</td>
<td>1.207</td>
<td>1.000</td>
</tr>
<tr>
<td>gaussian</td>
<td>2.000</td>
<td>1.505</td>
</tr>
<tr>
<td>hanning</td>
<td>2.000</td>
<td>2.667</td>
</tr>
<tr>
<td>hamming</td>
<td>1.820</td>
<td>2.935</td>
</tr>
</tbody>
</table>

$^a$ $\Delta v$ is the frequency sampling interval.

\[
\Delta v_{FWHP}^{FB} = \Delta v \quad (D.2)
\]
\[
\Delta v_{FWHP}^{HC} = 2.000\Delta v \quad (D.3)
\]
\[
\Delta v_{SB}^{FB} = \Delta v \quad (D.4)
\]
\[
\Delta v_{SB}^{HC} = 2.667\Delta v \quad (D.5)
\]

**D.1 Function Integrals**

**D.1.1 Sinc**

\[
\Delta v_{SB}^{a} = \frac{\int_{-\infty}^{\infty} \sin^2 \left( \frac{\pi v}{\Delta v} \right) dv}{\int_{-\infty}^{\infty} \sin^2 \left( \frac{\pi v}{\Delta v} \right) dv} \quad (D.6)
\]

\[
= \frac{\left[ 2 \int_{0}^{\infty} \sin^2 \left( \frac{\pi v}{\Delta v} \right) dv \right]^2}{2 \int_{0}^{\infty} \sin^2 \left( \frac{\pi v}{\Delta v} \right) dv} \quad (D.6)
\]

\[
= \frac{\left[ 2 \left( \frac{\pi^2}{\Delta v} \right) \right]^2}{2 \pi \left( \frac{\pi^2}{\Delta v} \right)} \quad (D.6)
\]

\[
= 1 \quad (D.6)
\]

**D.1.2 Gaussian**

\[
\Delta v_{SB}^{a} = \frac{\left\{ \int_{-\infty}^{\infty} \exp(- \ln(2) v^2) dv \right\}^2}{\int_{-\infty}^{\infty} \exp(- \ln(2) v^2) dv} \quad (D.7)
\]

\[
= \frac{\left\{ 2 \int_{0}^{\infty} \exp(- \ln(2) v^2) dv \right\}^2}{2 \int_{0}^{\infty} \exp(- \ln(2) v^2) dv} \quad (D.7)
\]

\[
= \frac{\pi}{\ln(2)} \quad \sqrt{\frac{\pi}{2 \ln(2)}} \quad (D.7)
\]
\[
\Delta v_{SB} = \frac{\int_{-(N-1)}^{N-1} \left\{ \frac{1}{2} [1 + \cos(2\pi v)] \right\}^2 dv}{\int_{-(N-1)}^{N-1} \left\{ \frac{1}{2} [1 + \cos(2\pi v)] \right\}^2 dv} \quad (D.8)
\]

Normally, we take 3 channels and give them weights 0.25, 0.5, 0.25. Therefore, \( \Delta v_{SB} \) becomes

\[
\Delta v_{SB} = \frac{\int_{-2}^{2} \left\{ \frac{1}{2} [1 + \cos(2\pi v)] \right\}^2 dv}{\int_{-2}^{2} \left\{ \frac{1}{2} [1 + \cos(2\pi v)] \right\}^2 dv} = \frac{4}{3} = \frac{8}{3} \quad (D.9)
\]

### D.1.4 Hamming

The Hamming function is just the Hanning function with different weighting.

\[
\Delta v_{SB} = \frac{\int_{-(N-1)}^{N-1} [0.54 + 0.46 \cos(2\pi v)]^2 dv}{\int_{-(N-1)}^{N-1} [0.54 + 0.46 \cos(2\pi v)]^2 dv} \quad (D.10)
\]
\[
eq \frac{(2.16)^2}{1.5896} \\
\cong 2.935078
\]